XVI. *On the Expansion of Sea-water by Heat.* *By* T. E. THORPE, *Ph.D., and* A. W. RÜCKER, *M.A. (Fellow of Brasenose College, Oxford), Professors of Chemistry and Physics in the Yorkshire College of Science. Communicated by* W. B. CARPENTER, *M.D., LL.D., F.R.S.*

THE extensive contributions which have recently been made to the physical history of the ocean have shown the desirability of exact knowledge of the relations of sea-water to heat. We have accordingly thought it worth while to make observations in order to determine the law of the thermal expansion of sea-water.

Hitherto the most important attempt to solve this problem was made by the late Prof. HUBBARD, of the United States National Observatory; the results of his investigation are contained in MAURY's 'Sailing Directions,' 1858, vol. i. p. 237.

MUNCKE, nearly 50 years ago, determined the expansion of an artificial sea-water at various temperatures between 0° and 100° C.; but our confidence in the results as applicable to natural sea-water is affected by the circumstance that the solution was prepared from data furnished by the imperfect analyses of VOGEL and BOUILLON LA-GRANGE.

The observations of DESPRETZ were confined to temperatures below 13°·27, as the main object of his inquiry was to determine the point of maximum density of sea-water. The subsequent investigations of NEUMANN and ROSSETTI were equally limited, as they were undertaken with the same view.

In accordance with our instructions, Captain CAMPBELL, of the Anchor Liner 'Europa,' kindly obtained for us a number of samples of Atlantic-Ocean water at various periods during a voyage from New York to Glasgow. The particular sample selected for observation was collected in lat. 50° 48′ N. and long. 31° 14′ W. The temperature of the surface-water was 52°·5 F. In the remarks accompanying the specimen it is stated that "the weather had been moderate and pleasant for some days previously, and up to the morning of collection; since then have had a brisk S. gale with heavy cross sea." Two series of experiments have been made with this sample.

In the first series our method of observation was precisely the same as that already employed by one of us in determining the expansion of the liquid chlorides of phosphorus[*]. It was essentially that already used by KOPP and PIERRE; *i. e.* the expansion was observed in thermometer-shaped vessels (so-called *dilatometers*) graduated and accu-

[*] Proc. Roy. Soc. vol. xxiii. p. 364.

rately calibrated. The dilatometers were constructed of flint glass containing 33 per cent. of oxide of lead, and the coefficients of expansion were determined for the limits of temperature between which our observations were made by observing the increase in the apparent volume of mercury. The calibration was made by a method substantially the same as that employed by KOPP—that is, by determining the weights of mercury contained in the various parts of the graduated stems. In several particulars, however, the method actually adopted possessed some points of advantage over that upon which it was based, both in improved modes of keeping the temperature of the bath constant and in withdrawing the mercury, and also in the plan adopted of weighing the mercury withdrawn so as to avoid removing the instrument from the bath. The thermometer used was one of a set made by CASELLA; 1° on each instrument had a length of about 9 millims. They were previously compared with Kew Standards belonging to Sir W. THOMSON. Frequent determinations were made of the zero-point of the thermometer during the progress of the observations; indeed the alteration in the fixed point had been noted sufficiently often over a period of nearly six years to make it possible to represent it by means of an equation like that obtained under similar conditions by Prof. W. H. MILLER ("Determination of Standard Pound," Phil. Trans. for 1855). The bath in which the instruments were immersed contained from thirty to forty litres of water, which was agitated by a stirrer worked by a small hydraulic engine. Direct experiments have shown that the temperature of the whole mass was thus made sensibly uniform.

The bath was heated by steam driven in from a small metallic boiler. When the temperature at which an observation was to be made was nearly attained, the delivery of the steam was regulated so as to maintain the temperature as nearly as possible constant; and at least six observations were made with a very slowly rising and an equal number with a very slowly falling thermometer. The readings were obtained by the aid of a telescope furnished with a micrometer-eyepiece, and the result given in each case is the mean of at least six observations.

As we desired to make our observations on the sea-water in its natural condition, it was considered unnecessary to attempt to extend them over a wide range of temperature. It was found that the water might be heated to about 50° C. before the air in solution made its appearance, provided that the dilatometer was kept perfectly still; but from the unavoidable vibration produced by the action of the stirrers in the bath, minute bubbles were formed at four or five degrees below this point. Accordingly the observations have not been extended beyond 40°. For the purpose of reducing specific-gravity observations this limit is amply sufficient, since sea-water never attains so high a temperature even in shallow places within the tropics.

The specific gravity of the sea-water at 0° C., compared with distilled water at the same temperature, was found, by the bottle, to be 1·02867.

The results of the first series of observations are contained in Table I.

TABLE I.

Dilatometer B.   Thermometer A.

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	2828·56	2828·54	−0·02
3·34	2829·03	2829·03	0·00
6·54	2829·83	2829·84	+0·01
9·58	2830·89	2830·91	+0·02
12·75	2832·35	2832·30	−0·05
15·94	2833·95	2833·98	+0·03
19·15	2835·91	2835·93	+0·02
22·32	2838·00	2838·09	+0·09
25·45	2840·55	2840·44	−0·11
28·88	2843·27	2843·24	−0·03
32·08	2846·01	2846·05	+0·04
35·02	2848·78	2848·77	−0·01

These results may be represented with sufficient accuracy by a formula of the form

$$V = K + Lt + Mt^2 + Nt^3.$$

The values of the constants K, L, M, and N were found by substituting successively for V and t the 12 corresponding values given in the Table as deduced from the experiments, and the expressions so obtained were then added together in consecutive groups of three, so as to form 4 equations of condition by the solution of which the 4 unknown quantities were determined.

The values of K, L, M, and N were thus found to be—

$$K = 2828·540,$$
$$L = \quad 0·091311,$$
$$M = \quad 0·0171712,$$
$$N = - \ 0·00009373.$$

The third column in the foregoing Table gives the volumes, calculated by the aid of these values, which correspond to the observed temperatures contained in the first column; and the fourth column shows the differences between the observed and calculated results.

Dividing each coefficient by K, and correcting for the expansion of the glass (0·0000254 for 1° C.), we obtain the following formula, as representing the expansion between 0° and 36° of sea-water of sp. gr. 1·02867 at 0°, the volume at 0° being 1.

$$V = 1 + ·000057682 \ t + ·0000060715 \ t^2$$
$$- ·000000032983 \ t^3.$$

The variations in the specific gravity of oceanic waters are, comparatively speaking, very small; according to the observations of Mr. J. Y. BUCHANAN, the Chemist to the 'Challenger' Expedition, they lie between the extremes 1·0278 and 1·0240. Still the question suggests itself—Is the law of the thermal expansion, as deduced from the foregoing observations, strictly applicable to all oceanic waters? The importance of an answer to this question will be evident when we bear in mind that, in order to be of value in the

investigation of the physical condition of the ocean, the results of the observations of the specific gravity of sea-water must be correct to one in the fourth decimal place. Of course we should have reason to expect that the variations in the rate of expansion of sea-water as it usually occurs, although doubtless appreciable, must be very small; nevertheless we thought it desirable to attempt to determine the relation between the salinity of the water, as indicated by the specific gravity, and its thermal expansion. Accordingly we constructed dilatometers of greatly increased capacity; and in order to expose a larger surface to the source of heat the bulbs were made in the form of elongated cylinders instead of being spherical, as in the case of the instrument we had hitherto employed. The new instruments were calibrated with great care, and the determination of the rates of expansion of the glass was made between the limits of temperature to which they were to be used. An additional set of thermometers was made for us by CASELLA; they are denoted in the succeeding Tables by α and β. Thermometer α was graduated for temperatures between $-9°$ and $29°$; the length of $1°$ was 13 millims.; the range of β was from $21°$ to $67°$, and the length of $1°$ was about 11 millims. Both instruments could therefore be easily read to $0°\cdot01$.

As it was desirable to obtain data to cover all possible cases of dilution and concentration in naturally occurring sea-water, we diluted quantities of our sea-water with distilled water so as to have samples of, approximately, the specific gravities $1\cdot020$ and $1\cdot025$; and we concentrated a third quantity by evaporation until its specific gravity was increased to about $1\cdot033$. We also redetermined the expansion of the original sea-water by means of the new dilatometers. Two perfectly independent series of observations were made with each solution, different dilatometers and thermometers being used in each case.

The details of the observations are exhibited in Tables II.–IX.; the first column in each gives the reduced and corrected temperatures, and the second the reduced volume (uncorrected for expansion of glass) of the liquid in the dilatometer. Formulæ expressing these results were obtained by the same method as that applied to the first set of observations as described on p. 407; and as in some of the series of experiments more than twelve observations were made, while in no case was a larger number used in deducing the formulæ, we have in the Tables marked the observations not employed for that purpose with an asterisk. The formulæ are given at the head of the Tables containing the observations to which they refer; the volumes calculated by their aid for the temperatures in the first columns are given in the third columns, the differences between the observed and calculated values being exhibited in the fourth columns. The specific gravities were taken at $0°$, and are referred to distilled water of the same temperature.

TABLE II.

Sample A.

Determination of Specific Gravity:—

Experiment I. . . . . 1·03298

„ II. . . . . 1·03305

Mean . . . . 1·033015

Series I. Dilatometer O. Thermometer A.

$$V = 6179·487 + 0·38985\ t + 0·0294072\ t^2$$
$$- 0·000075624\ t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
7·07	6183·81	6183·69	− 0·12
10·10	6186·30	6186·35	+ 0·05
12·90	6189·17	6189·25	+ 0·08
16·85	6193·97	6194·04	+ 0·07
19·38	6197·59	6197·54	− 0·05
22·29	6201·97	6201·95	− 0·02
24·99	6206·42	6206·41	− 0·01
26·90	6209·75	6209·78	+ 0·03
30·04	6215·71	6215·69	− 0·02
32·89	6221·33	6221·43	+ 0·10
35·74	6227·65	6227·53	− 0·12
39·67	6236·49	6236·51	+ 0·02

TABLE III.

Sample A.   Series II.

Dilatometer P.   Thermometers α and β.

$$V = 6177·933 + 0·31536\ t + 0·0335818\ t^2$$
$$- 0·000151131\ t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	6178·02	6177·93	− 0·09
1·99	6178·64	6178·69	+ 0·05
4·31	6179·87	6179·90	+ 0·03
6·34	6181·23	6181·24	+ 0·01
9·02	6183·39	6183·40	+ 0·01
12·08	6186·40	6186·38	− 0·02
*15·25	6189·98	6190·02	+ 0·04
18·18	6193·88	6193·86	− 0·02
20·86	6197·83	6197·75	− 0·08
23·87	6202·44	6202·54	+ 0·10
*24·06 β	6202·80	6202·86	+ 0·06
27·91	6209·57	6209·61	+ 0·04
30·25	6214·03	6214·02	− 0·01
32·79	6219·08	6219·05	− 0·03

TABLE IV.

Sample B (undiluted sea-water).

Estimation of Specific Gravity:—

Experiment I. . . . . 1·02866

„   II. . . . . 1·02866

Mean . . . 1·02866

Series II.   Dilatometer P.   Thermometer A.

$$V = 6173 \cdot 037 + 0 \cdot 21316t + 0 \cdot 0365255t^2$$
$$- 0 \cdot 000185263t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
4·11	6174·52	6174·52	0·00
7·07	6176·34	6176·30	−0·04
10·08	6178·67	6178·70	+0·03
12·91	6181·43	6181·45	+0·02
16·85	6186·16	6186·11	−0·05
19·38	6189·54	6189·54	0·00
22·29	6193·94	6193·88	−0·06
25·00	6198·33	6198·30	−0·03
26·90	6201·51	6201·59	+0·08
30·04	6207·40	6207·38	−0·02
32·90	6212·93	6212·99	+0·06
35·74	6218·90	6218·85	−0·05
39·67	6227·42	6227·41	−0·01

TABLE V.

Sample B.   Series III.

Dilatometer O.   Thermometers α and β.

$$V = 6192 \cdot 135 + 0 \cdot 17498t + 0 \cdot 0379567t^2$$
$$- 0 \cdot 000207014t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	6192·13	6192·14	+0·01
1·99	6192·63	6192·63	0·00
4·30	6193·58	6193·57	−0·01
6·34	6194·71	6194·72	+0·01
9·02	6196·65	6196·65	0·00
12·08	6199·43	6199·42	−0·01
*15·25	6202·94	6202·90	−0·04
18·17	6206·54	6206·60	+0·06
20·86	6210·57	6210·42	−0·15
23·87	6215·04	6215·12	+0·08
*24·07 β	6215·40	6215·45	+0·05
27·91	6221·98	6222·08	+0·10
30·26	6226·27	6226·45	+0·18
32·78	6231·65	6231·36	−0·29

TABLE VI.

Sample C.

Determination of Specific Gravity:—

Experiment I. . . . . 1·02494

„ II. . . . . 1·02489

Mean . . . 1·024915

Series I. Dilatometer O. Thermometers α and β.

$$V = 6193 \cdot 603 + 0 \cdot 10714t + 0 \cdot 0381813t^2$$
$$- 0 \cdot 000191475t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	6193·68	6193·60	−0·08
3·24	6194·29	6194·34	+0·05
6·45	6195·81	6195·83	+0·02
9·65	6198·00	6198·02	+0·02
12·77	6200·76	6200·80	+0·04
15·84	6204·18	6204·12	−0·06
18·88	6208·04	6207·95	−0·09
21·98	6212·45	6212·37	−0·08
24·98	6216·95	6217·12	+0·17
28·15 β	6222·50	6222·60	+0·10
31·13	6228·09	6228·16	+0·07
34·31	6234·67	6234·49	−0·18
*36·71	6239·64	6239·52	−0·12

TABLE VII.

Sample C. Series II.

Dilatometer P. Thermometer A.

$$V = 6178 \cdot 490 + 0 \cdot 09990t + 0 \cdot 0392112t^2$$
$$- 0 \cdot 000220711t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	6178·39	6178·49	+0·10
2·95	6179·21	6179·12	−0·09
4·51	6179·73	6179·74	+0·01
7·50	6181·36	6181·35	−0·01
10·41	6183·48	6183·53	+0·05
15·00	6188·11	6188·07	−0·04
17·35	6190·95	6190·87	−0·08
20·95	6195·70	6195·76	+0·06
23·83	6200·14	6200·15	+0·01
26·60	6204·64	6204·73	+0·09
29·84	6210·51	6210·52	+0·01
32·91	6216·49	6216·38	−0·11

TABLE VIII.

Sample D.

Determination of Specific Gravity :—

Experiment I. . . . . 1·02079

    ,,   II. . . . . 1·02072

                           Mean . . . 1·020755

Series I.   Dilatometer P.   Thermometers α and β.

$$V = 6177 \cdot 987 + 0 \cdot 02119 t + 0 \cdot 0442784 t^2$$
$$- 0 \cdot 000313329 t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	6177·89	6177·99	+0·10
3·24	6178·40	6178·37	−0·03
6·45	6179·68	6179·61	−0·07
9·64	6181·62	6181·62	0·00
12·79	6184·33	6184·30	−0·03
15·85	6187·50	6187·53	+0·03
18·87	6191·41	6191·25	−0·16
21·98	6195·54	6195·59	+0·05
24·98	6200·09	6200·20	+0·11
*28·13 β	6205·25	6205·46	+0·21
31·12	6210·65	6210·77	+0·12
34·32	6217·07	6216·76	−0·31
36·71	6222·06	6221·38	−0·68

TABLE IX.

Sample D.   Series II.

Dilatometer O.   Thermometer A.

$$V = 6188 \cdot 329 + 0 \cdot 030506 t + 0 \cdot 03823208 t^2$$
$$- 0 \cdot 000171645 t^3.$$

Temp. C.	Obs. vol.	Calc. vol.	Diff.
0·00	6188·23	6188·33	+0·10
2·94	6188·78	6188·74	−0·04
4·50	6189·28	6189·22	−0·06
7·50	6190·65	6190·64	−0·01
10·42	6192·63	6192·60	−0·03
14·99	6196·75	6196·80	+0·05
17·35	6199·49	6199·47	−0·02
20·96	6204·10	6204·18	+0·08
23·83	6208·50	6208·44	−0·06
26·59	6212·90	6212·94	+0·04
29·84	6218·72	6218·72	0·00
32·92	6224·68	6224·64	−0·04

The numbers in the fourth columns in each of the preceding Tables being satisfactorily small, the next step was to compare the expressions obtained from the different series of experiments on the same solution. For this purpose L, M, and N in each of the formulæ were divided by the corresponding values of K, and series of equations were thus obtained of the form

$$v = 1 + l't + m't^2 + n't^3,$$

giving the apparent expansion for each solution.

These equations were then corrected for the alterations in the volumes of the dilatometers due to the changes of temperature, the coefficients of expansion being taken to be

$$\text{for dilatometer O} \quad . \quad . \quad . \quad . \quad . \quad 0 \cdot 00002459,$$
$$\text{,,} \qquad \text{,,} \qquad \text{P} \quad . \quad . \quad . \quad . \quad . \quad 0 \cdot 00002382$$

for 1° C. being between 0° and 40°. The resulting expressions, of which

$$v = 1 + lt + mt^2 + nt^3$$

may be taken as the type, give the true expansions of the solutions as deduced from the separate series of experiments.

The values of l, m, and n are given in Table X.

<div align="center">TABLE X.</div>

Sample.	Series.	l.	m.	n.
A. {	I.	·000087677	·0000047604	$-$·000000012121
	II.	·000074867	·0000054370	$-$·000000024334
B. {	I.	·000057682	·0000060715	$-$·000000032983
	II.	·000058352	·0000059171	$-$·000000029871
	III.	·000052849	·0000061299	$-$·000000033281
C. {	I.	·000041889	·0000061650	$-$·000000030763
	II.	·000039989	·0000063468	$-$·000000035571
D. {	I.	·000020391	·0000071663	$-$·000000050546
	II.	·000029520	·0000061782	$-$·000000027585

The following Table shows the volumes calculated for a few temperatures by the aid of these constants.

TABLE XI.

$t°$ C.	Sample A.		Sample B.			Sample C.		Sample D.		$t°$ C.
	Series I.	Series II.	Series I.	Series II.	Series III.	Series I.	Series II.	Series I.	Series II.	
0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	0
2	19	17	14	14	13	11	11	7	8	2
4	43	39	32	32	31	26	26	19	21	4
6	69	64	56	56	53	47	46	37	39	6
8	100	93	83	83	80	71	71	60	62	8
10	134	127	115	115	111	100	100	87	89	10
12	171	164	151	150	146	134	133	119	120	12
15	237	226	212	211	206	191	191	175	174	15
18	305	297	281	279	274	257	257	239	237	18
21	383	375	358	356	351	332	331	312	309	21
24	468	459	443	440	434	413	412	392	388	24
27	559	551	533	530	524	502	501	478	476	27
30	659	648	630	627	621	598	595	570	570	30
33	764	752	733	730	722	699	695	666	671	33
36	876	861	841	838	830	806	801	766	778	36

The agreement between the numbers referring to the same solution in these Tables may be considered satisfactory; the greatest differences, which are observed in the case of sample A, may in part be due to the fact that, unfortunately, in Series I. no observations could be made at temperatures below 7°; but as we have adopted the means of the different equations obtained for each series as the true expressions of the expansions, and as these formulæ are, as we shall see hereafter, connected together by a simple relation, which that referring to A obeys on the whole satisfactorily, we have not thought it necessary to repeat the experiments, as those already obtained prove sufficiently accurate for our purpose when thus checked.

The mean values of the constants given in Table X. are exhibited in Table XII., and are the coefficients of the powers of t in the formulæ we adopt as expressing in each case the relation between the volumes and the temperatures of the solutions submitted to experiment, and to which we shall hereafter refer as the *mean* formulæ.

TABLE XII.

	l.	m.	n.
A.	·000081272	·0000050987	—·000000018227
B.	·000056294	·0000060395	—·000000032045
C.	·000040939	·0000062559	—·000000033167
D.	·000024955	·0000066723	—·000000039065

Expressions for the expansions of the individual solutions having been thus obtained, it only remained to investigate a formula embracing them all as particular cases of a relation between the temperature, volume, and salinity of sea-water as measured by its

specific gravity at 0°. Table XIII. was therefore prepared: it shows, side by side, for purposes of comparison, the volumes of the samples A, B, C, and D at a few temperatures, that at 0° C. being in each case supposed to be unity.

TABLE XIII.

$t°$.	A.	B.	C.	D.
6	1·00066	1·00055	1·00046	1·00038
10	131	114	100	88
15	232	210	191	174
21	379	355	331	310
27	555	529	501	477
33	758	728	697	668

The differences between consecutive columns in this Table multiplied by 10^5 give us the following numbers:—

$t°$.	A−B.	B−C.	C−D.
6	11	9	8
10	17	14	12
15	22	19	17
21	24	24	21
27	26	27	24
33	30	31	29

Dividing the numbers in each of these columns by those in that which follows it, we obtain the quotients (correct to one place of decimals)—

$$\frac{A-B}{B-C}=1·2 \quad 1·2 \quad 1·1 \quad 1·0 \quad 1·0 \quad 1·0$$

$$\frac{B-C}{C-D}=1·1 \quad 1·2 \quad 1·1 \quad 1·1 \quad 1·1 \quad 1·1$$

The numbers in these two groups are, to within the limits of the error of experiment, constant.

Accordingly we may assume that if v, v', and v'' are the volumes at the same temperature of quantities of the solutions, the volume of each of which at 0° C. is unity, the value of $\frac{v-v'}{v'-v''}$ is independent of the temperature, and is therefore, if the solutions to which v' and v'' refer are supposed to be always the same, a function of s (the specific gravity of the third solution at 0° C.) only, so that

$$\frac{v-v'}{v'-v''}=f(s). \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

Secondly, noticing that the numbers in each of the above groups are approximately equal to each other and to unity, and bearing in mind the fact that the differences

3 M 2

between the specific gravities of the solutions employed, which are

$$\cdot004355, \quad \cdot003748, \quad \text{and} \quad \cdot004160$$

respectively, differ in the fourth place of decimals, we are led to the conclusion that $f(s)$ is a linear function of s, and that differences in the fourth place of decimals in the value of the specific gravities do not involve important changes in the law of thermal expansion.

From this it follows that equation (1), when written at length, is of the form

$$v = 1 + bt + ct^2 + dt^3 + (b't + c't^2 + d't^3)(p + qs), \quad \ldots \ldots \ldots (2)$$

or

$$v = \varphi(t) + \psi(t)f(s)*.$$

As this expression is of no theoretical value, it is unnecessary to describe, step by step, the arithmetical operations by which the constants were determined; it is sufficient to say that a larger number of equations than that required for the purpose may be obtained by equating the coefficients of t in (2) to their values as determined for the solutions submitted to experiment and given in Table XII., and that the numbers given by the equations of condition so obtained have been in some cases slightly modified. The values taken were :—

$$
\begin{aligned}
b &= \quad \cdot00008097, & b' &= -\cdot5509 \times 10^{-5}, \\
c &= \quad \cdot0000049036, & c' &= \quad \cdot020198 \times 10^{-5}, \\
d &= -\cdot000000012289, & d' &= -\cdot00033276 \times 10^{-5}.
\end{aligned}
$$

The expression $p + qs$ may be conveniently written in the form

$$11\cdot95 - 940\,(s - 1\cdot02),$$

in which it is unnecessary to express s to more than 4 places of decimals. Table XIV. shows the agreement between the volumes calculated by the mean formulæ and the general expression thus determined.

TABLE XIV.

$t°$ C.	Sample A.			Sample B.			Sample C.			Sample D.		
	Mean formula.	General formula.	Diff.	Mean formula.	General formula.	Diff.	Mean formula.	General formula.	Diff.	Mean formula.	General formula.	Diff.
0	1·00000	1·00000	0	1·00000	1·00000	0	1·00000	1·00000	0	1·00000	1·00000	0
2	18	18	0	14	14	0	11	10	−1	7	7	0
4	41	40	−1	32	33	+1	26	26	0	20	19	−1
6	66	66	0	55	56	+1	46	47	+1	38	36	−2
8	96	95	−1	82	83	+1	71	72	+1	61	58	−3
10	131	130	−1	114	115	+1	100	101	+1	88	86	−2
12	167	167	0	149	150	+1	133	135	+2	119	118	−1
15	232	229	−3	210	209	−1	191	191	0	174	173	−1
18	301	298	−3	278	277	−1	257	258	+1	238	237	−1
21	379	376	−3	355	353	−2	331	333	+2	310	310	0
24	463	461	−2	439	437	−2	412	415	+3	390	390	0
27	555	553	−2	529	527	−2	501	503	+2	477	477	0
30	653	652	−1	626	623	−3	596	597	+1	570	569	−1
33	758	758	0	728	726	−2	697	697	0	668	665	−3
36	869	870	+1	836	834	−2	803	802	−1	772	766	−6

* The numerical constants involved in the above formula are given in the forms in which they were, for facility of calculation, determined. The expression can, of course, be easily transformed to the simpler form, $V = F_1(t) + s\,F_2(t)$.

This Table amply verifies the accuracy of the general formula over the range of the experiments; and it has accordingly been employed to calculate Table XV., in which the results of the investigation are summed up.

The values of the functions of t on which the value of v depends were first determined for every integral value of t between $0°$ and $36°$, and the value of $f(s)$ was calculated when s was put equal to $1·02$ and $1·03$, and also for several intermediate values of the variable. The different numbers given by $\psi(t)$ were multiplied by the values of $f(s)$ in succession, and the results being added (algebraically) to the corresponding values of $\phi(t)$, gave the values of v for the selected temperatures and specific gravities.

The specific gravities at the different temperatures (σ) were then obtained by means of the formula

$$\sigma v = s.$$

Writing this in the form

$$\sigma = \frac{s}{\phi(t) + \psi(t)(p + qs)},$$

it is easy to show, by considering the different possible values of $\phi(t)$, $\psi(t)$, and s, that $\frac{d\sigma}{ds}$ is for the range of the observations so nearly constant that no error greater than $·00002$ could be introduced into the calculated specific gravities by the assumption that it is independent of s; and the values of σ calculated for the different selected values of s prove that this superior limit is really considerably in excess of the true error, and that therefore we may with safety assume that $\frac{d\sigma}{ds}$ is constant.

We are therefore able to give in Table XV. all the data necessary to calculate the specific gravity of sea-water of any degree of salinity at any temperature between $0°$ and $36°$. Column II. contains the specific gravities at the temperatures given in column I. of a solution the specific gravity of which at $0°$ C. is $1·02000$; column III. contains the numbers which must be subtracted from those in column I. for each increase of $1°·0$ over the temperatures opposite to which they are placed; and column IV. the numbers which must be added for each increase of $·00001$ of the specific gravity of the solutions at zero. At the head of columns III. and IV. are the numbers of ciphers which must be prefixed to the figures written in them in the unit place.

TABLE XV.

I. Temperature.	II. Specific Gravity.	III. Proportional parts for 1° C.	IV. Proportional parts for ·00001 increase in sp. gr.	I. Temperature.	II. Specific Gravity.	III. Proportional parts for 1° C.	IV. Proportional parts for ·00001 increase in sp. gr.
		·0000	·0000			·0000	·0000
0	1·02000	3	1	19	1·01740	25	0·944
1	1·01997	4	0·995	20	1·01715	25	0·943
2	1·01993	5	0·990	21	1·01690	26	0·941
3	1·01988	6	0·986	22	1·01664	27	0·940
4	1·01982	8	0·982	23	1·01637	28	0·938
5	1·01974	9	0·979	24	1·01609	29	0·937
6	1·01965	11	0·975	25	1·01580	29	0·935
7	1·01954	12	0·972	26	1·01551	30	0·934
8	1·01942	13	0·969	27	1·01521	30	0·932
9	1·01929	14	0·966	28	1·01491	31	0·930
10	1·01915	15	0·963	29	1·01460	32	0·928
11	1·01900	17	0·961	30	1·01428	32	0·925
12	1·01883	17	0·958	31	1·01396	32	0·922
13	1·01866	19	0·956	32	1·01364	33	0·919
14	1·01847	20	0·954	33	1·01331	33	0·915
15	1·01827	21	0·952	34	1·01298	33	0·912
16	1·01806	21	0·950	35	1·01265	34	0·908
17	1·01785	22	0·948	36	1·01231	34	0·903
18	1·01763	23	0·946				

In order to facilitate the use of the Table we subjoin directions for its application in the form of rules, and give a couple of examples.

Given the specific gravity of a sample of sea-water at any temperature t, to find it at 0° C.:—Look out in column I. the figure giving the number of entire degrees of the temperature; multiply the fraction (if any) by which the observed temperature exceeds that number by the corresponding number in III., and subtract the result from the corresponding number in column II. Subtract the difference from the observed specific gravity, and divide the number so obtained by that corresponding to the observed temperature in column IV. (without prefixing the ciphers at the top of the column); add the quotient to 1·02000, and the sum will be the specific gravity required.

Example I. Specific gravity observed at 18°·5 C.=1·02475. To find it at 0° C., number opposite 18 in column III. is ·00023, which multiplied by ·5 equals ·000115; and
$$1·017630 - ·000115 = 1·017515.$$
Subtract this from the observed specific gravity,
$$1·024750 - 1·017515 = ·007235.$$
Divide by ·945, and the quotient is ·00765, which added to 1·02000 gives 1·02765 as the specific gravity at 0° C.

Example II. Specific gravity observed at 15° =1·02570. To find it at 0° C.,
$$1·02570 - 1·01827 = ·00743,$$
and
$$\frac{·00743}{·952} = ·00780.$$
Therefore specific gravity at 0°=1·02000+·00780=1·02780.

Having thus linked together the results of our observations under one general law, it remains to compare our conclusions with those of Prof. HUBBARD, whose investigation is, as we have already stated, the only one which covers the same ground as our own.

If, as Mr. BUCHANAN points out, the specific gravities of different specimens of sea-water vary only between 1·0278 and 1·0240, the results to be of value must be correct to the fourth decimal place; obviously therefore the formulæ of reduction ought to be as accurate as the experiments themselves.

In MAURY's work Professor HUBBARD's Table of the thermal dilatation of sea-water extends from 22° F. to 200° F., the volume of water at 60° F. being taken as unity. In order to compare it with our own, we have transcribed the portion contained between 22° F. (−5°·5 C.) and 95° F. (36° C.), making the volume at 0° the unit, as in our own Table.

TABLE XVI.

° C.	HUBBARD.	T. & R.	° C.	HUBBARD.	T. & R.
−5·55	1·00012		+2·0	1·00009	1·00014
5·00	1·00006		4·0	25	33
4·44	1·00003		6·0	45	56
3·89	1·00000		8·0	71	83
3·33	0·99998		10·0	98	115
2·78	0·99997		12·0	132	150
2·22	0·99996		15·0	192	209
1·67	0·99996		18·0	265	277
1·11	0·99997		21·0	343	353
0·56	0·99998		24·0	430	437
0·00	1·00000	1·00000	27·0	526	527
			30·0	626	623

A glance at the two columns shows that the volumes do not agree to within the limits of error above indicated, the numbers being approximately the same for the higher and lower but widely divergent for intermediate temperatures. Mr. HUBBARD's observations were made by a method substantially the same as our own, but the results are affected by a circumstance which might at first sight be supposed to conduce to their accuracy. Mr. HUBBARD employed unusually large volumes of sea-water in his experiments; but it should be noted, as indeed KOPP has already pointed out, that in such observations the increased difficulty of obtaining a uniform temperature throughout the entire mass more than compensates for the accuracy arising from the greater displacement of the liquid column. The stems of Mr. HUBBARD's instruments were not graduated, the readings being taken from attached scales, probably of wood or metal. Moreover, in the reduction, the inaccurate coefficient of expansion of mercury given by DULONG and PETIT was employed. These circumstances may probably serve to account for the discrepancies between the two sets of observations. At all events we may point out that it would seem almost impossible that errors of the magnitude of the discrepancies we are discussing can occur in our own results. Taking as an example the particular case of 15° C., where the difference between the two values is large, and referring to Table XI.,

we see that the three values of the volume we obtained by three independent series of experiments were 1·00212, 1·00211, and 1·00206, and that the general formula gives 1·00209, which is, to within a fraction in the fifth place, equal to the mean of the three observations, and certainly cannot be affected with so great an error as 17 in the fifth place, which is, however, the difference between it and Prof. HUBBARD's number.

We may also point out that it appears from Prof. HUBBARD's Table that the point of maximum density of average sea-water is 28°·5 F. or −1°·94 C. If we compare this result with other observations on record, we find that it is probably between 2° and 3° C. too high. Excluding MUNCKE's result of −5°·25, which is probably too low, we find that DESPRETZ obtained −3°·67 (which is probably too high for average sea-water), NEUMANN −4°·74 in the case of a mixture of sea-water from Trieste, Genoa, and Heligoland, and ROSSETTI −3°·90 in that of Adriatic water of specific gravity 1·02814 at 0°. The number obtained from our general formula, viz. −5°·14, is probably too low by at least 0°·5 C.; but it is necessary to remember that the formulæ we give are based on observations between 0° and 36°, and are only applicable to temperatures between these limits.

It is quite evident from the foregoing observations that the laws of the thermal expansion of all ocean sea-waters cannot be assumed to be sensibly the same. The difference at 15° between the volumes of sea-waters of specific gravities 1·024 and 1·0278 is 18 in the fifth place; at 30° it is 26; so that even the use of the formula corresponding to the mean of the two numbers would, for high temperatures and relatively weak or strong solutions, introduce errors in the fourth place of decimals, whence it follows that the alterations in the law of thermal expansion depending on the salinity of the solutions cannot be neglected.